

# Symmetric functions and random partitions

Andrei Okounkov\*

## Abstract

These are notes from the lectures I gave at the NATO ASI “Symmetric Functions 2001” at the Isaac Newton Institute in Cambridge (June 25 – July 6, 2001). Their goal is an informal introduction to asymptotic combinatorics related to partitions.

These notes are based on the three lectures that I gave in Cambridge in July of 2001. Some parts, especially the computations, are given here in more detail than in the actual lectures; other parts, such as connections to geometry of the moduli spaces of curves, have been omitted. Our goal in these lectures was to give an informal introduction to the results contained in [5, 28, 29] (there are also many closely related papers such as [6, 10, 12, 15, 16, 17, 19]), as well as to the general subject of asymptotic combinatorics related to partitions. No attempt, however, will be made here to survey this already quite large and very rapidly growing subject. Many further references to the literature will be pointed out in text.

I want to take this opportunity to thank many people who made this NATO ASI in Cambridge such a wonderful and memorable meeting: the organizers, the sponsors, the lecturers, and, of course, the participants. There is also a very long list of people who, in addition to their vast influence on the field itself, influenced my understanding of it through numerous fruitful discussion, in particular: P. Deift, P. Diaconis, S. Fomin, A. Its, K. Johansson, R. Kenyon, R. Stanley, C. Tracy, A. Vershik, H. Widom, and many others, including, certainly, my collaborators A. Borodin, G. Olshanski, and N. Reshetikhin.

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\*Department of Mathematics, University of California at Berkeley, Evans Hall #3840, Berkeley, CA 94720-3840. E-mail: okounkov@math.berkeley.edu. Partial financial support by NSF grant DMS-0096246, Sloan foundation, and the Packard foundation is gratefully acknowledged.

It is very tragic that one of the pioneers of the field and, also, one of the organizers of this meeting, Sergei Kerov, did not live to participate in it. Most of the material in these lectures is directly related to his groundbreaking and lasting work and I want to dedicate these lecture to his memory.

# 1 Portrait of a large random partition

## 1.1

A partition is a nonincreasing sequence

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0)$$

of nonnegative integers such that  $\lambda_i = 0$  for all sufficiently large  $i$ . Similarly, a plane partition is a matrix of nonnegative integers with nonincreasing rows and columns and finitely many nonzero entries, for example

$$\pi = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 \\ 3 & 2 & 1 & \\ 2 & 1 & & \end{pmatrix}. \quad (1)$$

A standard geometric image associated to a partition is its diagram. There are several competing traditions of drawing the diagrams of partitions; we follow the one illustrated in Figure 3 which shows the diagram of the partition  $(8, 5, 4, 2, 2, 1)$ .

Similarly, the diagram of a plane partition is a 3-dimensional object which looks like a stack of cubes pushed into a corner. The 3D diagram corresponding to the partition (1) is shown in Figure 1

## 1.2

Naturally, partitions are inseparable from symmetric functions. Similarly, application of symmetric functions to enumeration of plane partitions is a combinatorial classics, an illustrated account of which can be found, for example, in [8].

Our goal in these lectures is to explain some recent progress in applying symmetric function theory to get rather refined information about random

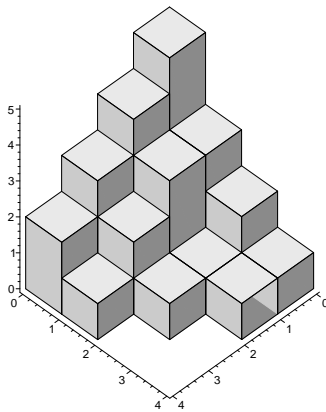


Figure 1: A 3-dimensional diagram  $\pi$

partitions, both ordinary and 3-dimensional. Random partitions, in addition to being interesting simply for their own sake, arise in a large variety of other contexts.

One of the most important situations in which random partitions arise is the study of increasing and decreasing subsequences in a random permutation, see for example [1, 13]. Namely, the celebrated Robinson-Schensted correspondence associates to any permutation  $g \in S(N)$  a partition  $\lambda_{RS}(g)$  of  $N$ , which by Greene-Fomin extension (see e.g. [7]) of Schensted's theorem [32] encodes the increasing and decreasing subsequences of  $g$ . Concretely, the number

$$\lambda_{RS}(g)_1 + \cdots + \lambda_{RS}(g)_k$$

of squares in the first  $k$  rows of  $\lambda_{RS}(g)$  equals the maximal cardinality of a union of  $k$  increasing subsequences of  $g$ . The same is true about columns of  $\lambda_{RS}$  and decreasing subsequences of  $g$ .

Increasing subsequences of random permutations play a central role, for example, in the analysis of many random growth processes. Therefore, one wants to understand the push-forward of the uniform probability measure on  $S(N)$  under the map  $g \mapsto \lambda_{RS}(g)$ . This push-forward is well known to be the *Plancherel measure* on partitions of  $N$  given by

$$\mathbb{P}_{\text{Pl}}(\lambda) = \frac{(\dim \lambda)^2}{N!}, \quad |\lambda| = N,$$

where  $\dim \lambda$  is the dimension of the corresponding irreducible representation

of the symmetric group.

The number  $\dim \lambda$  has a very transparent combinatorial meaning: it is the number of chains of partitions (also known as standard tableaux) of the form

$$\lambda = \lambda^{(0)} \searrow \lambda^{(1)} \searrow \lambda^{(2)} \searrow \dots \searrow \lambda^{(N)} = \emptyset, \quad (2)$$

where  $\mu \searrow \nu$  means that  $\nu$  is obtained from  $\mu$  by removing a square. There are several useful formulas for the number  $\dim \lambda$ , in particular, the hook formula. The textbooks [31, 34] are a wonderful place to learn about these things.

### 1.3

The following law of large numbers for the Plancherel measure was discovered by Logan and Shepp [23] and, independently, Vershik and Kerov [38, 39]. Consider a diagram of a partition of  $N$  as in Figure 3 and scale it in both directions by factor of  $\sqrt{N}$  (note that the rescaled diagram has unit area). The rescaled diagram for a Plancherel-typical partition of a large number  $N$  will then look something like the diagram in Figure 2. Now look at the

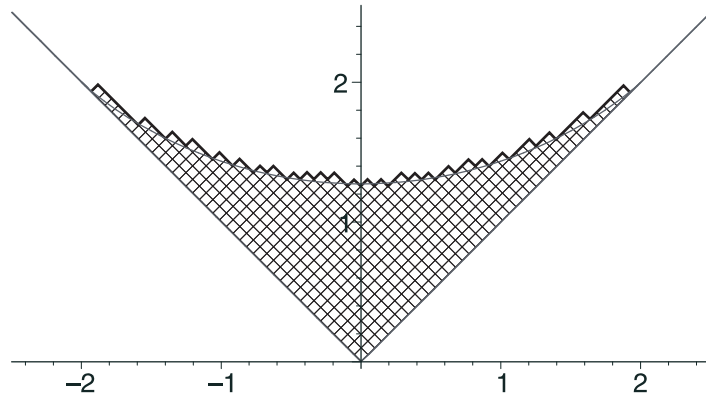


Figure 2: The limit shape of a Plancherel-typical diagram.

zigzag path in Figure 2 formed by the boundary of the diagram. This zigzag is a graph of a piecewise linear function and, in this way, the Plancherel measure becomes a measure on continuous (and even Lipschitz with constant 1) functions. The discovery of Logan-Shepp-Vershik-Kerov is that, as  $N \rightarrow \infty$ , this measure converges (see [23, 38, 39] for precise statements) to the

delta measure supported on the following function

$$\Omega(u) = \begin{cases} \frac{2}{\pi} (u \arcsin(u/2) + \sqrt{4 - u^2}), & |u| \leq 2, \\ |u|, & |u| > 2. \end{cases}$$

This formula looks, at first, rather formidable, but its derivative is much simpler, namely

$$\Omega'(u) = \frac{2}{\pi} \arcsin \frac{u}{2}, \quad (3)$$

naturally extended by  $\pm 1$  for  $u \notin [-2, 2]$ .

In fact, the derivative  $\Omega'(u)$  is a very natural object because it measures how often the zigzag path in Figure 2 goes up (respectively, down) in the neighborhood of a given point on the boundary of the limit shape. In other words, if we think of the zigzag path as a random sequence of “ups” and “downs”,  $\Omega'(u)$  is related to the probability to observe an “up” in a given position. Traditionally, probabilities of this kind are called *correlation functions*. More precisely, the the probability to observe “up” in a given position is the 1-point correlation function of our random sequence of “ups” and “downs”.

## 1.4

It is now natural to ask about higher correlations, for example, what is the probability to observe several consecutive “ups”, or, more generally, any given pattern of “ups” and “downs”?

Such questions can be conveniently formalized using the following set

$$\mathfrak{S}(\lambda) = \{\lambda_i - i + \tfrac{1}{2}\} \subset \mathbb{Z} + \tfrac{1}{2}. \quad (4)$$

This is the set of “downs” of the boundary of  $\lambda$  as illustrated in Figure 3.

We remark that the map  $\lambda \mapsto \mathfrak{S}(\lambda)$  is a bijection of the set of partitions and the set of subsets  $S \subset \mathbb{Z} + \frac{1}{2}$  such that

$$\left| S \setminus (\mathbb{Z} + \tfrac{1}{2})_{<0} \right| = \left| (\mathbb{Z} + \tfrac{1}{2})_{<0} \setminus S \right| < \infty.$$

In other words, the set  $\mathfrak{S}(\lambda)$  has equally many positive elements and negative “holes” (this number is the length of the diagonal in the diagram  $\lambda$ ).

We now take a finite set

$$X = X(N) = \{x_1, \dots, x_n\} \subset \mathbb{Z} + \tfrac{1}{2},$$

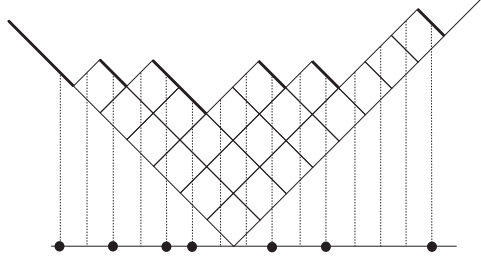


Figure 3: Geometric meaning of  $\mathfrak{S}(\lambda) = \{\bullet\}$

which is allowed to depend on  $N$  in such a way that

$$\frac{x_i}{\sqrt{N}} \rightarrow u_i \in [-2, 2], \quad N \rightarrow \infty,$$

and all differences  $x_i - x_j$  have a limit, finite or infinite. The set  $X$  is the set where we want to observe the “downs”. The parameters  $u_i$  describes the global position of our observations on the limit shape  $\Omega$  while the differences  $x_i - x_j$  record their relative local separation.

It is clear the following probabilities (a.k.a. correlation functions)

$$\mathbb{P}_{\text{Pl}}(X \subset \mathfrak{S}(\lambda)) = \frac{1}{N!} \sum_{X \subset \mathfrak{S}(\lambda)} (\dim \lambda)^2$$

to observe “downs” in positions  $X = \{x_1, \dots, x_n\}$  include all possible information about the local shape of the random partition  $\lambda$ . As it turns out, all these probabilities have a very pretty  $N \rightarrow \infty$  limit, namely, we have the following:

**Theorem 1 ([5]).** *For  $X$  as above we have*

$$\mathbb{P}_{\text{Pl}}(X \subset \mathfrak{S}(\lambda)) \rightarrow \det \left[ \mathbb{K}_{\sin}(x_i - x_j; \phi_i) \right]_{i,j=1 \dots n}, \quad (5)$$

as  $N \rightarrow \infty$ , where

$$\mathbb{K}_{\sin}(x; \phi) = \frac{\sin \phi x}{\pi x}, \quad k \in \mathbb{Z}, \quad (6)$$

is the discrete sine kernel with parameters

$$\phi_i = \arccos(u_i/2), \quad i = 1, \dots, n. \quad (7)$$

In the formula (6), it is understood that

$$\mathbb{K}_{\text{sin}}(\infty; \phi) = 0,$$

which implies that the determinant in (5) factors into blocks corresponding to distinct values of  $u_i$ . In other words, it means that the occurrences of “downs” in different parts of the limit shape become independent in the  $N \rightarrow \infty$  limit, which, intuitively, is what one expects.

It is also understood in (6) that

$$\mathbb{K}_{\text{sin}}(0; \phi) = \frac{\phi}{\pi}.$$

In particular, the 1-point correlation function, that is, the density of “downs” is

$$\mathbb{P}_{\text{Pl}}(\{x\} \subset \mathfrak{S}(\lambda)) \rightarrow \frac{\arccos u/2}{\pi}, \quad \frac{x}{\sqrt{N}} \rightarrow u.$$

This translates into (3) and also explains the role of the parameter  $\phi$  in the sine kernel: it is the density parameter.

Finally, it is natural to extend the function (7) by 0 or  $\pi$  for  $u \notin [-2, 2]$ . Theorem 1 remains valid and says simply that for  $u > 2$  (resp.  $u < -2$ ) the zigzag in Figure 2 goes only up (resp. down).

## 1.5

One encounters a formula which looks just like the formula (5) in the random matrix theory, see e.g. [14, 25]. Concretely, consider a Gaussian probability measure  $\mathbb{P}_H$  on the real vector space of Hermitian  $N \times N$  matrices  $H$  with density  $e^{-\text{tr } H^2}$ . The spectrum of  $\sigma(H)$  of a random matrix  $H$  is a random  $N$ -element subset of  $\mathbb{R}$ . Consider the probability density  $\rho_H(X)$  to have  $X \subset \sigma(H)$  for some fixed  $X = \{x_1, \dots, x_n\}$ , that is, the probability density to observe eigenvalues of  $H$  in given positions. Formally, this probability density is defined as follows

$$\rho_H(x_1, \dots, x_n) dx_1 \cdots dx_n = \mathbb{P}_H \left( \left| \sigma(H) \cap \bigcup_{i=1}^n [x_i, x_i + dx_i] \right| = n \right).$$

In other words,  $\rho_H(x_1, \dots, x_n) dx_1 \cdots dx_n$  is actually a measure whose values on a given set  $A \subset \mathbb{R}^n$  is related to the probability of finding  $n$  eigenvalues of  $H$  in  $A$ .

It is important to remember that since  $\rho_H(X)$  is a density, it transforms nontrivially under a change of variables and, in particular, it scales nontrivially under the scaling of variables.

The 1-point correlation function  $\rho_H(x_1)$  describes the density of eigenvalues of  $H$ . The analog of the limit shape  $\Omega$  in this case is the Wigner semicircle law

$$\frac{1}{\sqrt{N}} \rho_H(x) \rightarrow \frac{\sqrt{4-u^2}}{2\pi}, \quad \frac{x}{\sqrt{N}} \rightarrow u \in [-2, 2], \quad N \rightarrow \infty.$$

In other words, scaled by  $\sqrt{N}$ , the density of eigenvalues converges to the semicircle with diameter  $[-2, 2]$ . The normalization  $\frac{1}{\sqrt{N}}$  is related to the fact that there are  $N$  eigenvalues of  $H$  and, because they all are concentrated on the segment  $[-2\sqrt{N}, 2\sqrt{N}]$ , the density of eigenvalues is expected to be of order  $\sqrt{N}$ .

By the same token, the typical spacing between the eigenvalues is of order  $1/\sqrt{N}$ . It is then meaningful to ask about the asymptotics of  $\rho(X)$  as

$$\frac{x_i}{\sqrt{N}} \rightarrow u \in [-2, 2], \quad \sqrt{N}(x_i - x_j) \rightarrow y_i - y_j,$$

for some fixed  $y_i \in \mathbb{R}$ . One finds (see [14, 25]) that in this limit

$$\frac{\rho(X)}{N^{|X|/2}} \rightarrow \det \left[ \mathbb{K}_{\sin}(y_i - y_j; r(u)) \right], \quad r(u) = \sqrt{1 - (u/2)^2}. \quad (8)$$

This looks almost identical to (5), with one minor and one not so minor difference. The minor difference is a different dependence of the density parameter in the sine kernel on the global position  $u$ . A more important difference is that in (5) the variables  $x_i - x_j$  take only discrete values, whereas in (8) we deal with continuous variables.

In the continuous case, we can get rid of the density parameter in the sine kernel by a simple rescaling. After this transformation, the global question of eigenvalue density and the local question of spacings between the eigenvalues become completely decoupled. Much work has been done in the random matrix theory to show that the eigenvalue spacing have the same sine kernel form for many much more general measures on the space of matrices, in other words, to show that the occurrence of the sine kernel is *universal*, see e.g. [14, 18].

Comparing, (5) with (8) reveals an striking “universality of formulas”, namely that the formulas are the same even when the meaning is different.



This parallel, however, is more than purely formal. For example, the discrete sine kernel process shares many features of its continuous counterpart, such as the repulsion of eigenvalues which we will discuss momentarily. Also, continuous distributions, such as the distributions of eigenvalues of a  $N \times N$  Gaussian Hermitian matrix, can be obtained as a suitable limit of certain natural measures (of the kind considered below) on partitions with  $N$  parts.

## 1.6

It can be inferred from the formula (8) that the eigenvalues of a random matrix have a strong tendency to be equally spaced, much more so than simply a random collection of points on a line. For example, the 2-point correlation function (in the unit density case) is equal to

$$\det \begin{bmatrix} 1 & \frac{\sin \pi(x-y)}{\pi(x-y)} \\ \frac{\sin \pi(y-x)}{\pi(y-x)} & 1 \end{bmatrix} = 1 - \left( \frac{\sin \pi(x-y)}{\pi(x-y)} \right)^2 ,$$

which is very small (has a double zero) near  $x - y = 0$  and has maxima for  $x - y = 1, 2, 3, \dots$

It was noted by Kerov that there exists a similar repulsion of “ups” and “downs” in Figure 2. Namely, the “ups” and “downs” tend to be much more uniformly spaced than outcomes of independent trials, such as, for example, the occurrences of “heads” in independent coin tosses. Let us illustrate this by a numerical example. Suppose that we are in the very middle of the limit shape (that is,  $u = 0$ ), where the density of both “ups” and “downs” is equal to

$$\left. \frac{\arccos(u/2)}{\pi} \right|_{u=0} = \frac{1}{2} .$$

In this case the discrete sine kernel takes an especially simple form

$$\mathbb{K}_{\sin}(x; \frac{\pi}{2}) = \frac{\sin(\pi x/2)}{\pi x} = \begin{cases} \frac{1}{2}, & x = 0, \\ \frac{(-1)^k}{\pi x}, & x = 2k + 1, \\ 0, & x = \pm 2, \pm 4, \dots \end{cases}$$

Consider the following probabilities

$$\begin{aligned} a_n &= \mathbb{P}_{\sin}(\text{up, down, up, down, } \dots, \text{up, down}) , \\ b_n &= \mathbb{P}_{\sin}(\text{up, up, } \dots, \text{up, down, down, } \dots, \text{down}) , \end{aligned}$$

where each string has length  $2n$ . These discrete sine kernel probabilities are given by certain  $(2n) \times (2n)$  determinants. If the “ups” and “downs” were like “heads” and “tails” of independent coin tosses, both  $a_n$  and  $b_n$  would equal  $2^{-2n}$ . But as the plot of  $\log_2 a_n$  and  $\log_2 b_n$  in Figure 4 shows

$$a_n \gg 2^{-2n} \gg b_n,$$

which means that the discrete sine kernel process very strongly prefers even spacing of its “ups” and “downs”.

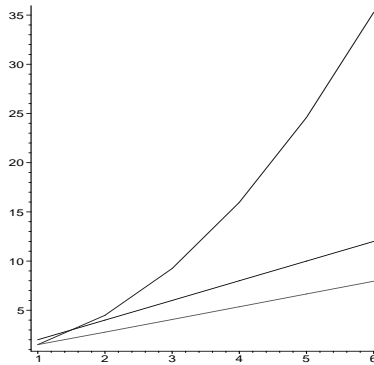


Figure 4: The plot (from top to bottom) of  $-\log_2 b_n$ ,  $2n$ , and  $-\log_2 a_n$

## 1.7

The connection between the Plancherel measure and random matrices becomes exact near the edge of the Logan-Shepp-Vershik-Kerov limit shape and the Wigner semicircle, respectively. In other words, the maximal parts  $\lambda_1, \lambda_2, \dots$  of a Plancherel random partition, suitably scaled, behave in  $N \rightarrow \infty$  limit *exactly* like the maximal eigenvalues of an Hermitian random matrix. For  $\lambda_1$  it was proven by Baik, Deift, and Johansson in [2], who also conjectured that the same holds for any  $\lambda_i$ ,  $i = 2, 3, 4, \dots$  (for  $i = 2$  they verified this in [3]).

By Schensted’s theorem, mentioned in Section 1.2, this means that the longest increasing subsequence of a random permutation is, asymptotically, distributed like the largest eigenvalue of a Hermitian random matrix. This distribution is known as the Tracy-Widom distribution [35] and is given explicitly in term of a certain solution of the Painlevé II equation.

The random point process describing the maximal eigenvalues of a random Hermitian matrix  $H$  is known as the *Airy ensemble* because the correlation function of this process are described by the same determinantal formula as in (8) with the sine kernel replaced by the Airy kernel

$$\mathbb{K}_{\text{Airy}}(x, y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}. \quad (9)$$

Here  $\text{Ai}(x)$  is the Airy function

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{is^3/3 + ixs} ds = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{s^3}{3} + sx\right) ds. \quad (10)$$

The joint distributions of the largest, 2nd largest, and so on points of the Airy ensemble satisfy quite complicated nonlinear PDEs, generalizing the Painlevé II equation for the Tracy-Widom distribution. The description of the Airy ensemble by its correlation functions is much simpler and this is the description that we will use in what follows.

## 1.8

The first proof of the full Baik-Deift-Johansson conjecture was given in [27] by establishing a direct asymptotics correspondence between permutations and matrices via a certain correspondence between counting maps on surfaces and counting branched covering of the sphere. A different proof was soon after found, independently, in [5] and [17]. It is this second proof that will be explained in these lectures.

It is clear from Theorem 1 that all correlation become trivial at  $u = 2$ , which is because near  $u = 2$  the “downs” become so rare that they occur infinitely far away from each other. In other words, for a Plancherel typical partition  $\lambda_i - \lambda_j \rightarrow \infty$  as  $N \rightarrow \infty$  for any fixed  $i \neq j$ . In fact,  $\lambda_i - \lambda_j$  should be typically of order  $N^{1/6}$ , which can be seen heuristically as follows.

The expected number of  $\lambda_i$ ’s that are larger than  $2\sqrt{N} - C$  is equal to

$$\begin{aligned} \mathbb{E} \left| \mathfrak{S}(\lambda) \cap [2\sqrt{N} - C, \infty) \right| &\sim \sqrt{N} \int_{2-C/\sqrt{N}}^2 \frac{\arccos(u/2)}{\pi} du \\ &\sim \frac{2C^{3/2}}{3\pi N^{1/4}} \end{aligned}$$

for  $C \ll \sqrt{N}$ . Since there are exactly  $k$  parts of  $\lambda$  that are greater or equal to  $\lambda_k$  we expect that for  $k \ll \sqrt{N}$

$$k \approx \text{const} \frac{(2\sqrt{N} - \lambda_k)^{3/2}}{N^{1/4}},$$

whence

$$2\sqrt{N} - \lambda_k \approx \text{const} k^{2/3} N^{1/6}$$

which precisely means that  $\lambda_i - \lambda_j$  should be typically of order  $N^{1/6}$  for large  $N$ .

This heuristics explains the scaling in the following result (conjectured by Baik, Deift, and Johansson), which is the analog of Theorem 1 for the edge  $u = 2$  of the limit shape  $\Omega$ .

**Theorem 2 ([27, 5, 17]).** *As  $N \rightarrow \infty$ , the random variables*

$$\frac{\lambda_i - 2\sqrt{N}}{N^{1/6}}, \quad i = 1, 2, 3, \dots,$$

*converge in joint distribution to the Airy ensemble.*

## 1.9

We now move on to plane partitions and 3D diagrams. On this set, we consider the following measure:

$$\mathbb{P}_{3D}(\pi) = \frac{1}{Z_{3D}} q^{|\pi|}, \quad q \in [0, 1),$$

where  $|\pi|$  is the volume of a 3D diagram  $\pi$  and  $Z_{3D}$  is the normalizing factor

$$Z_{3D} = \sum_{\pi} q^{|\pi|} = \prod_{n>0} (1 - q^n)^{-n},$$

the second equality being the classical identity due to McMahon.

In particular, this measure gives equal weight  $q^N$  to all partitions of size  $N$ . It is natural to ask why we consider this essentially uniform measure and not some analog of the Plancherel measure on 3D diagrams (which could involve, for example, the number of ways to construct  $\pi$  starting from the empty partition by adding a square at a time). The answer is that this

measure *is* the proper analog of the Plancherel measure for the following reason. By definition and (2),  $\mathbb{P}_{\text{Pl}}(\lambda)$  is proportional to the number of chains of the form

$$\emptyset = \lambda^{(-N)} \nearrow \dots \nearrow \lambda^{(-1)} \nearrow \lambda^{(0)} \searrow \lambda^{(1)} \searrow \dots \searrow \lambda^{(N)} = \emptyset. \quad (11)$$

such that

$$\lambda^{(0)} = \lambda.$$

If one assembles all these  $\lambda^{(i)}$  into an array of partitions, one sees a structure closely resembling a 3D diagram and, by construction, the Plancherel measure is the distribution of the central slice of this structure induced by the uniform measure on all chains of the form (11).

This also suggests that the proper way to look at 3D diagram is to separate them into their diagonal slices (and also suggest a common generalization of the measures  $\mathbb{P}_{\text{Pl}}$  and  $\mathbb{P}_{3\text{D}}$  to be introduced later). In particular, the diagonal slices of the partition from (1) are

$$(2) \prec (3, 1) \prec (4, 2) \prec (5, 3, 1) \succ (3, 1) \succ (2, 1) \succ (1),$$

where  $\lambda \succ \mu$  means that  $\lambda$  and  $\mu$  interlace (this relation replaces the relation  $\lambda \searrow \mu$  in (11)). The proper modification of the definition (4) for the 3D case is

$$\mathfrak{S}_{3D}(\pi) = \left\{ \left( j - i, \pi_{ij} - \frac{i+j-1}{2} \right) \right\} \subset \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}, \quad (12)$$

which has the following geometric meaning. The set  $\mathfrak{S}_{3D}(\pi) \subset \mathbb{R}^2$  is formed by the centers of the horizontal faces of a diagram  $\pi$  under the following mapping

$$\mathbb{R}^3 \ni (x, y, z) \mapsto (y - x, z - \frac{x+y}{2}) \in \mathbb{R}^2.$$

This is illustrated in Figure 5 which shows the projections of the horizontal faces of the 3D diagram from Figure 1.

## 1.10

The analog of the  $N \rightarrow \infty$  limit for the Plancherel measure is the  $q \rightarrow 1$  limit of  $\mathbb{P}_{3\text{D}}$ . Indeed, as  $q \rightarrow 1$ , larger and larger diagrams become important. It can be shown (see e.g. [29]) that as  $q \rightarrow 1$

$$r^3 |\pi| \rightarrow 2\zeta(3)$$

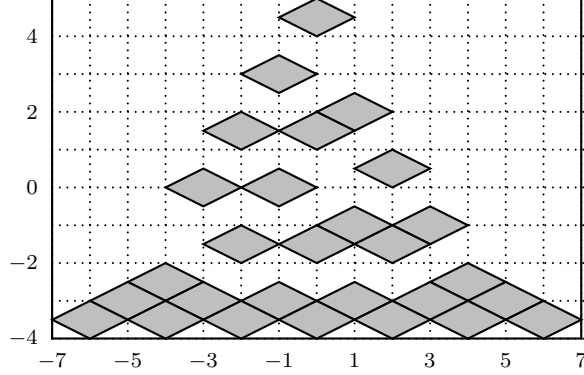


Figure 5: Projections of horizontal faces of the diagram from Figure 1

in probability, where

$$r = -\ln q.$$

is a convenient parameter that goes to  $+0$  as  $q \rightarrow 1$ .

Thus, it is very natural to ask what is the asymptotics of the correlation functions

$$\mathbb{P}_{3D}(X \subset \mathfrak{S}_{3D}(\pi)) , \quad X = (t_i, h_i) \subset \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z},$$

as  $q \rightarrow 1$  and the set  $X$  is allowed to vary in such a way that

$$r(t_i, h_i) \rightarrow (u, v) \in \mathbb{R}^2,$$

while all differences

$$(t_i, h_i) - (t_j, h_j) \in \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$$

have a limit, finite or infinite.

At this point, it should not come as a big surprise that there is an analog of Theorem 1 for 3D diagrams that answers this question. To state it, we need the following 2-dimensional analog of the sine kernel. One way to write the sine kernel is like this:

$$\mathbb{K}_{\sin}(k; \phi) = \frac{\sin k\phi}{\pi k} = \frac{1}{2\pi i} \int_{e^{-i\phi}}^{e^{i\phi}} \frac{dw}{w^{k+1}}.$$

A 2-dimensional analog of this is the following *incomplete beta kernel*

$$\mathbb{K}_{\text{beta}}(t, h; z) = \frac{1}{2\pi i} \int_{\bar{z}}^z (1-w)^t \frac{dw}{w^{h+t/2+1}}, \quad (13)$$

where  $z \in \mathbb{C}$  is a parameter that replaces the density parameter  $\phi \in \mathbb{R}$ . The path of the integration in (13) crosses the segment  $(0, 1)$  for  $t \geq 0$  and the segment  $(-\infty, 0)$  for  $t < 0$ .

Now the analog of Theorem 1 for 3D diagrams is the following. By symmetry, we can assume, without loss of generality that  $u \geq 0$ .

**Theorem 3 ([29]).** *For  $X$  as above we have*

$$\mathbb{P}_{3D}(X \subset \mathfrak{S}_{3D}(\pi)) \rightarrow \det \left[ \mathbb{K}_{\text{beta}}(t_i - t_j, h_i - h_j; z) \right]_{i,j=1 \dots n}, \quad (14)$$

as  $q \rightarrow 1$ , where the parameter  $z$  is the point of intersection of two circles

$$\{z, \bar{z}\} = \{|z| = e^{-u/2}\} \cap \{|z - 1| = e^{-u/4 - v/2}\}, \quad \Im z > 0,$$

or else, in case these circles do not intersect,  $z = \bar{z}$  is the point of  $\{|z| = e^{-u/2}\}$  which is the closest to  $\{|z - 1| = e^{-u/4 - v/2}\}$ .

### 1.11

In particular, the limit of the 1-point correlation function, that is, the limiting density of the horizontal faces, is given by

$$\mathbb{K}_{\text{beta}}(0, 0; z) = \frac{\arg z}{\pi} = \frac{1}{\pi} \arccos \left( \cosh \frac{u}{2} - \frac{e^{-v}}{2} \right). \quad (15)$$

Here the second equality is literally true if the two circles in Theorem 3 intersect. Otherwise, the arccosine has to be extended by 0 or  $\pi$  which, just like for the Plancherel measure, corresponds to trivial correlations. In other words, when the two circles in Theorem 3 do not intersect it means that we are trying to measure correlations outside of the limit shape.

This limit shape, the existence of which was first established by Vershik [37] and which was described explicitly by Cerf and Kenyon in [10], can be reconstructed by integrating the density (15). This integral can be evaluated in terms of the dilogarithm function. The plot of the limit shape can be seen in Figure 6.

## 2 Schur process and its correlations

### 2.1

It is difficult to deny that the best way to solve an asymptotic problem is to have a nice exact formula valid before the limit, which would make passing to

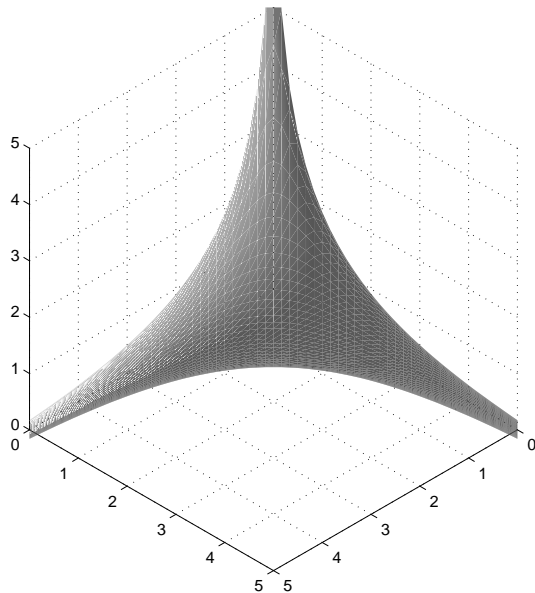


Figure 6: The limit shape for 3D diagrams

the limit conceptually clear, if not completely straightforward. A wonderful feature of our problems is that such exact formulas are indeed available.

In fact, exact formulas for correlation functions are available for much more general class of measures. These measures, which we call *Schur process*, will be described presently. By definition [29], the Schur process is a measure on sequences of partitions  $\{\lambda(t)\}$ ,  $t \in \mathbb{Z}$ , such that  $\lambda(t) = \emptyset$  for all but finitely many  $t$ . The probability of a configuration  $\{\lambda(t)\}$  is

$$\mathbb{P}_{\text{Schur}}(\{\lambda(t)\}) = \frac{1}{Z} \prod_{m \in \mathbb{Z} + \frac{1}{2}} s_{\lambda(m \pm \frac{1}{2}) / \lambda(m \mp \frac{1}{2})}(A_m), \quad (16)$$

where  $s_{\lambda/\mu}$  is a skew Schur function and  $A_m$  is some set of variables. The factor  $Z$  in (16) is introduced, as usual, for normalization, that is, to make  $\mathbb{P}_{\text{Schur}}$  a probability measure. The alphabets  $A_m$  and the choice of signs in (16) are the parameters of the process. In our examples, the choice of signs will always be understood to be such that

$$\dots \subset \lambda(-2) \subset \lambda(-1) \subset \lambda(0) \supset \lambda(1) \supset \lambda(2) \supset \dots$$



In other words, we would like our partitions  $\lambda(t)$  to grow up to the middle and then decay, just like in (11). The general formalism of Schur processes, however, works equally well for any choice of signs. The above definition of a Schur process is slightly different from the one given in [29], which avoids the awkward necessity of choosing signs, but is essentially equivalent.

## 2.2

For example, let

$$A_m = \{q^{|m|}\}, \quad m \in \mathbb{Z} + \frac{1}{2},$$

that is, let the alphabet  $A_m$  consist of a single variable  $q^{|m|}$ , where  $q$  is a parameter (e.g. a complex number such that  $|q| < 1$ ). It is a basic fact that

$$s_{\lambda/\mu}(t, 0, 0, \dots) = \begin{cases} t^{|\lambda/\mu|}, & \lambda \succ \mu, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lambda \succ \mu$  means that  $\lambda$  and  $\mu$  interlace, which can be also expressed as  $\lambda/\mu$  being a horizontal strip. Therefore, in this case the Schur process is supported on the sequences of the form

$$\dots \prec \lambda(-2) \prec \lambda(-1) \prec \lambda(0) \succ \lambda(1) \succ \lambda(2) \succ \dots$$

and the probability of such a configuration is proportional to

$$q^{\sum_{m \in \mathbb{Z} + \frac{1}{2}} |m| |\lambda(m \pm \frac{1}{2}) / \lambda(m \mp \frac{1}{2})|} = q^{\sum |\lambda(m)|}.$$

We recognize the measure  $\mathbb{P}_{3D}$ , where the diagram  $\pi$  is given by the sequence  $\{\lambda(m)\}$  of its diagonal slices.

The general formalism of [28, 29], which will be explained on an example later, gives the following exact formula for the correlation functions of the measure  $\mathbb{P}_{3D}$ . We have

$$\mathbb{P}_{3D}(X \subset \mathfrak{S}_{3D}(\pi)) = \det \left[ \mathbb{K}_{3D}((t_i, h_i), (t_j, h_j); q) \right]_{i,j=1 \dots n}, \quad (17)$$

where the kernel  $\mathbb{K}_{3D}$  is given by the following contour integral

$$\begin{aligned} \mathbb{K}_{3D}((t_i, h_i), (t_j, h_j); q) = \\ \frac{1}{(2\pi i)^2} \int_{|z|=1 \pm \epsilon} \int_{|w|=1 \mp \epsilon} \frac{1}{z-w} \frac{\Phi_{3D}(t_1, z)}{\Phi_{3D}(t_2, w)} \frac{dz dw}{z^{h_1 + \frac{|t_1|}{2} + \frac{1}{2}} w^{-h_2 - \frac{|t_2|}{2} + \frac{1}{2}}}. \end{aligned} \quad (18)$$

In this integral, one choses the top sign if  $t_1 \geq t_2$  and the bottom sign otherwise. The function  $\Phi_{3D}$  here is the following product

$$\Phi_{3D}(t, z) = \frac{\prod_{m > \max(0, -t)} (1 - q^m/z)}{\prod_{m > \max(0, t)} (1 - q^m z)}, \quad m \in \mathbb{Z} + \frac{1}{2}.$$

At first, the integral in (18) looks a little scary, but in fact it is not and, in particular, its asymptotic analysis, leading to Theorem 3 is more or less straightforward saddle point analysis, see [29] and below.

### 2.3

As another example, take the following specialization of the Schur process. Let  $A_m$  be empty for  $m \neq \pm \frac{1}{2}$  and for  $m = \pm \frac{1}{2}$  take the following specialization of the algebra of the symmetric functions

$$p_1 \mapsto \sqrt{\xi}, \quad p_k \mapsto 0, \quad k > 1,$$

where  $\xi > 0$  is a parameter. It is well known that this has the following effect

$$s_\lambda \left( A_{\pm \frac{1}{2}} \right) = \xi^{|\lambda|/2} \frac{\dim \lambda}{|\lambda|!}.$$

Thus, in this specialization, the Schur process becomes a measure on sequences

$$\dots \subset \emptyset \subset \emptyset \subset \lambda \supset \emptyset \supset \emptyset \supset \dots$$

where the distribution of the unique nontrivial partition in the middle is given by

$$\mathbb{P}_{PP}(\lambda) = \frac{1}{Z_{PP}} \xi^{|\lambda|} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2, \quad Z_{PP} = e^\xi. \quad (19)$$

This measure is the so-called Poissonization of the Plancherel measure, that is, the mixture of Plancherel measures on partitions of  $N$  for different values of  $N$  by means of the Poisson distribution on  $N$  with parameter  $\xi$

$$\mathbb{P}_{\text{Poisson}}(N) = \frac{1}{e^\xi} \frac{\xi^N}{N!}, \quad N = 0, 1, 2, \dots \quad (20)$$

In particular, from (20) we can infer the value of the normalization constant in (19).

It is a well known principle (known under various names such as e.g. *equivalence of ensembles*) that the  $N \rightarrow \infty$  asymptotics of the Plancherel measures should be equivalent to the  $\xi \rightarrow \infty$  limit of their Poissonizations (19). Essentially, this is because the distribution (20) has mean  $\xi$  and standard deviation  $\sqrt{\xi}$ , so for  $\xi \sim N \rightarrow \infty$  it is concentrated around  $N$ .

## 2.4

The general formula of [28, 29] specializes to the following exact formula for the correlations of the measure  $\mathbb{P}_{\text{PP}}$  (which is a limit case of the main formula of [6] and can be found as stated in [5] and [17])

$$\mathbb{P}_{\text{PP}}(X \subset \mathfrak{S}(\lambda)) = \det \left[ \mathbb{K}_{\text{Bessel}}(x_i, x_j; \xi) \right]_{i,j=1 \dots n}, \quad (21)$$

where  $\mathbb{K}_{\text{Bessel}}$  is the discrete Bessel kernel given by

$$\mathbb{K}_{\text{Bessel}}(x, y; \xi) = \frac{1}{(2\pi i)^2} \iint_{|z| > |w|} \frac{e^{\sqrt{\xi}(z - z^{-1} - w + w^{-1})}}{z - w} \frac{dz dw}{z^{x+\frac{1}{2}} w^{-y+\frac{1}{2}}} \quad (22)$$

$$= \sqrt{\xi} \frac{J_{x-\frac{1}{2}} J_{y+\frac{1}{2}} - J_{x+\frac{1}{2}} J_{y-\frac{1}{2}}}{x - y}. \quad (23)$$

Here  $J_n = J_n(2\sqrt{\xi})$  is the Bessel function of integral order  $n$  (recall that  $x, y \in \mathbb{Z} + \frac{1}{2}$ ) defined as the coefficient in the following Laurent series

$$e^{\frac{x}{2}(z - z^{-1})} = \sum_{n \in \mathbb{Z}} J_n(x) z^n.$$

The equivalence of (22) and (23) can be seen as follows. We have

$$\frac{1}{z^{x+\frac{1}{2}} w^{-y+\frac{1}{2}}} = -\frac{1}{x-y} \left( z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w} + 1 \right) \frac{1}{z^{x+\frac{1}{2}} w^{-y+\frac{1}{2}}},$$

which we can substitute in (22) and then integrate by parts using

$$\left( z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w} + 1 \right) \frac{e^{\sqrt{\xi}(z - z^{-1} - w + w^{-1})}}{z - w} = \sqrt{\xi} \left( 1 - \frac{1}{zw} \right) e^{\sqrt{\xi}(z - z^{-1} - w + w^{-1})}.$$

This yields (23).

## 2.5

We will now explain, on the example of the formula (21), the general formalism for obtaining such formulas developed in [28, 29]. Consider the following operator in the algebra of symmetric functions

$$\Gamma \cdot f = e^{\sqrt{\xi} p_1} f.$$

Since

$$p_1 s_\mu = \sum_{\lambda \searrow \mu} s_\lambda, \quad (24)$$

we conclude that

$$\Gamma \cdot 1 = \sum_{\lambda} \xi^{|\lambda|/2} \frac{\dim \lambda}{|\lambda|!} s_\lambda.$$

Consider the following projection operator

$$\delta_X \cdot s_\lambda = \begin{cases} s_\lambda, & X \subset \mathfrak{S}(\lambda) \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

Then, by construction

$$\mathbb{P}_{\text{PP}}(X \subset \mathfrak{S}(\lambda)) = \frac{(\Gamma^* \delta_X \Gamma \cdot 1, 1)}{(\Gamma^* \Gamma \cdot 1, 1)}, \quad (26)$$

where we use the standard Hall inner product on the algebra of symmetric functions

$$(s_\lambda, s_\mu) = \delta_{\lambda, \mu},$$

and where

$$\Gamma^* = \exp \left( \sqrt{\xi} \frac{\partial}{\partial p_1} \right),$$

is the adjoint of  $\Gamma$  with respect to this inner product. The relation

$$\left[ \frac{\partial}{\partial p_1}, p_1 \right] = 1$$

implies that

$$\Gamma^* \Gamma = e^\xi \Gamma \Gamma^*, \quad (27)$$

which together with

$$\Gamma^* \cdot 1 = 1 \quad (28)$$

gives another computation of the normalization constant

$$Z_{PP} = (\Gamma^* \Gamma \cdot 1, 1) = e^\xi .$$

Our goal is to apply a similar trick to evaluate the numerator in (26). For this we need a good realization of the operator (25). This is naturally provided by the fermionic Fock space formalism (see, e.g. [20]) which will be discussed presently.

## 2.6

Let the vector space  $V$  be spanned by  $e_k$ ,  $k \in \mathbb{Z} + \frac{1}{2}$ . The infinite wedge space  $\Lambda^{\frac{\infty}{2}} V$  is, by definition, spanned by vectors

$$v_S = e_{s_1} \wedge e_{s_2} \wedge e_{s_3} \wedge \dots ,$$

where  $S = \{s_1 > s_2 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$  is such a subset that both sets

$$S_+ = S \setminus (\mathbb{Z}_{\leq 0} - \frac{1}{2}) , \quad S_- = (\mathbb{Z}_{\leq 0} - \frac{1}{2}) \setminus S$$

are finite. We equip  $\Lambda^{\frac{\infty}{2}} V$  with the inner product in which the basis  $\{v_S\}$  is orthonormal. In particular, we have the vectors

$$v_\lambda = e_{\lambda_1 - \frac{1}{2}} \wedge e_{\lambda_2 - \frac{3}{2}} \wedge e_{\lambda_3 - \frac{5}{2}} \wedge \dots ,$$

where  $\lambda$  is a partition. The vector

$$v_\emptyset = e_{-\frac{1}{2}} \wedge e_{-\frac{3}{2}} \wedge e_{-\frac{5}{2}} \wedge \dots$$

is called the vacuum vector.

By construction, the  $\mathbb{C}$ -linear map

$$s_\lambda \rightarrow v_\lambda$$

is an isometric embedding of the algebra of symmetric functions into  $\Lambda^{\frac{\infty}{2}} V$ . It follows from (24) that the operator of multiplication by  $p_1$  is given in this realization by the natural action on  $\Lambda^{\frac{\infty}{2}} V$  of the following matrix

$$p_1 \mapsto \alpha = [\delta_{k,l-1}]_{k,l \in \mathbb{Z} + \frac{1}{2}} = \begin{bmatrix} \ddots & \ddots & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ & & & & & & \ddots & \ddots \end{bmatrix} \in \mathfrak{gl}(V) .$$

As we shall see in a moment, the operator (25) is given in this realization by something equally natural.

Consider the action of the matrix unit

$$E_{ii} = [\delta_{k,i} \delta_{l,i}]_{k,l \in \mathbb{Z} + \frac{1}{2}} = \begin{bmatrix} \ddots & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \in \mathfrak{gl}(V).$$

It is clear that

$$E_{ii} v_\lambda = \begin{cases} v_\lambda, & i \in \mathfrak{S}(\lambda) \\ 0, & \text{otherwise,} \end{cases}$$

and hence

$$\delta_X = \prod_{x \in X} E_{xx}.$$

This is already very nice, but we can actually do better by introducing the fermionic operators  $\psi_i$  which are, in some sense, the square roots of the operators  $E_{ii}$ .

By definition, the operator  $\psi_i$  is the wedging with the vector  $e_i$

$$\psi_i v = e_i \wedge v,$$

and the operator  $\psi_i^*$  is the adjoint operator with respect to the inner product on  $\Lambda^{\frac{\infty}{2}} V$ . One checks from definitions that

$$E_{ii} = \psi_i \psi_i^*,$$

and hence the formula (26) becomes

$$\mathbb{P}_{\text{PP}}(X \subset \mathfrak{S}(\lambda)) = \frac{1}{e^\xi} \left( e^{\sqrt{\xi} \alpha^*} \prod_{x \in X} \psi_x \psi_x^* e^{\sqrt{\xi} \alpha} \cdot v_\emptyset, v_\emptyset \right). \quad (29)$$

## 2.7

Now we want to exploit (27) and (28). By definition, set

$$\Psi_x = \text{Ad} \left( e^{\sqrt{\xi} \alpha^*} e^{-\sqrt{\xi} \alpha} \right) \cdot \psi_x, \quad (30)$$

and similarly define  $\Psi_x^*$ , where  $\text{Ad}$  denotes the action by conjugation. Note that since the operators  $\alpha$  and  $\alpha^*$  commute up to a central element, the ordering of these operators inside the  $\text{Ad}$  sign is immaterial. Combining (27), (28), and (30) with (29) gives the following

$$\mathbb{P}_{\text{PP}}(X \subset \mathfrak{S}(\lambda)) = \left( \prod_{x \in X} \Psi_x \Psi_x^* \cdot v_\emptyset, v_\emptyset \right). \quad (31)$$

It is now a pleasant combinatorial exercise, known as the Wick lemma, to verify that

$$\left( \prod_{x \in X} \Psi_x \Psi_x^* \cdot v_\emptyset, v_\emptyset \right) = \det [(\Psi_x \Psi_y^* \cdot v_\emptyset, v_\emptyset)]_{x, y \in X}.$$

This reduces the verification of (21) to proving that

$$(\Psi_x \Psi_y^* \cdot v_\emptyset, v_\emptyset) = \mathbb{K}_{\text{Bessel}}(x, y; \xi). \quad (32)$$

Since (22) gives, essentially, the generation function for the kernel  $\mathbb{K}_{\text{Bessel}}$ , it is natural to introduce similar generation functions

$$\begin{aligned} \psi(z) &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} z^k \psi_k, \\ \psi^*(w) &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} w^{-k} \psi_k^*. \end{aligned}$$

One then checks directly that

$$\Psi(z) = \text{Ad} \left( e^{\sqrt{\xi} \alpha^*} e^{-\sqrt{\xi} \alpha} \right) \cdot \psi(z) = e^{\sqrt{\xi}(z - z^{-1})} \psi(z),$$

which together with

$$(\psi(z) \psi^*(w) \cdot v_\emptyset, v_\emptyset) = \sum_{k = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots} z^k w^{-k} = \frac{\sqrt{zw}}{z - w}, \quad |z| > |w|$$

implies that

$$(\Psi(z) \Psi^*(w) \cdot v_\emptyset, v_\emptyset) = \frac{\sqrt{zw}}{z - w} e^{\sqrt{\xi}(z - z^{-1} - w + w^{-1})},$$

which proves (32) and completes the proof of (21).

## 3 Asymptotics

### 3.1

The integral representations of the type (18) and (22) are particularly suitable for asymptotic analysis by the classical method of steepest descent (see e.g. [4, 26]). The essence of this method is that the asymptotics of the integral

$$\int e^{MS(x)} dx, \quad M \rightarrow \infty, \quad (33)$$

for a real function  $S(x)$  is dominated by arbitrarily small neighborhoods of the points where the function  $S(x)$  takes its maximal value. If  $x_0$  is such a point then  $S'(x_0) = 0$  and hence

$$S(x) = S(x_0) + \frac{S''(x_0)}{2}(x - x_0)^2 + O((x - x_0)^3).$$

Assume that  $S''(x_0) \neq 0$  (and hence  $S''(x_0) < 0$ ) and introduce a new  $t$  variable by

$$x - x_0 = \frac{t}{\sqrt{-MS''(x_0)}}.$$

Then for fixed  $t$

$$MS(x) = MS(x_0) - \frac{t^2}{2} + o(1)$$

and we see that the leading contribution of  $x_0$  to (33) is given by

$$e^{MS(x_0)} \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi} \frac{e^{MS(x_0)}}{\sqrt{-MS''(x_0)}}.$$

Similar considerations apply in the case when the maximum of  $S(x)$  at  $x_0$  is degenerate, that is, when  $S''(x_0) = 0$ , but, for example,  $S^{IV}(x_0) < 0$ .

For similar integrals in the complex plane

$$\int_{\gamma} e^{MS(z)} dz, \quad M \rightarrow \infty,$$

where  $\gamma$  is some contour, one first deforms the contour  $\gamma$  to make it pass through the critical points  $z_0$  of  $S(z)$  (also known as saddle points) in such a way that the real part of  $S(z)$  has a local maximum along  $\gamma$  at  $z_0$ . After that, one proceeds as with the integral (33).



A qualitative conclusion from this argument is that *such integrals never have finite limit as  $M \rightarrow \infty$* . In other words, whenever one expects some finite limit, such as, for example, we expect (6) to be the  $\xi \rightarrow \infty$  asymptotics of (22), the contributions of saddle points computed by the method above will be simply zero. This makes one wonder how one can ever get some finite limit, because, after all, (6) is the  $\xi \rightarrow \infty$  asymptotics of (22) as we will see in a moment. The answer is that, as we deform the contours as required by the steepest descent method, we will be picking up residues, and those will have a nontrivial finite limit.

## 3.2

We now illustrate how this works on a particular example. Namely, let us prove that as

$$\frac{x}{\sqrt{\xi}}, \frac{y}{\sqrt{\xi}} \rightarrow u \in (-2, 2),$$

in such a way that  $x - y$  remains fixed, we have

$$\mathbb{K}_{\text{Bessel}}(x, y; \xi) \rightarrow \mathbb{K}_{\text{sin}}(x - y; \phi), \quad \phi = \arccos(u/2). \quad (34)$$

We have

$$\mathbb{K}_{\text{Bessel}}(x, y; \xi) = \frac{1}{(2\pi i)^2} \iint_{|z| > |w|} \frac{\exp\left(\sqrt{\xi} \left[S\left(z; \frac{x}{\sqrt{\xi}}\right) - S\left(w; \frac{y}{\sqrt{\xi}}\right)\right]\right)}{z - w} \frac{dz dw}{\sqrt{zw}}, \quad (35)$$

where

$$S(z; u) = z - z^{-1} - u \ln z.$$

The critical points of the function  $S(z, u)$  are the roots of

$$zS'(z; u) = z + z^{-1} - u \quad (36)$$

which for  $u \in (-2, 2)$  are clearly given by

$$z = e^{\pm i\phi}.$$

It is easy to see that the function  $\Re S(z; u)$  vanishes on the unit circle  $|z| = 1$  and the direction of its gradient, plotted in Figure 7, is easy to infer from (36).

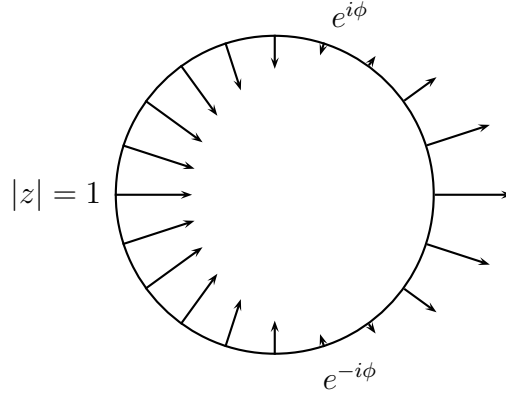


Figure 7: Gradient of  $\Re S(z)$  on the unit circle

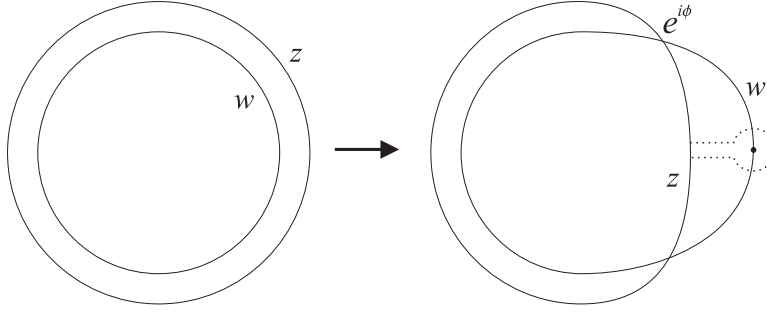


Figure 8: Deformation of the contours for the integral (35)

Thus, by the method of steepest descent we have to deform the contours of integration as shown in Figure 8, after which the integral, as explained above, will go to 0 as  $\xi \rightarrow \infty$ . However, as we deform the contours, we will pick up the residue at  $z = w$ , as is also shown in Figure 8. It is obvious that the residue of the integrand in (35) at  $z = w$  is equal to  $\frac{1}{2\pi i} \frac{dw}{w^{x-y+1}}$ . Thus, we conclude that

$$\mathbb{K}_{\text{Bessel}}(x, y; \xi) \rightarrow \frac{1}{2\pi i} \int_{e^{-i\phi}}^{e^{i\phi}} \frac{dw}{w^{x-y+1}} = \mathbb{K}_{\text{sin}}(x - y; \phi),$$

as was to be shown.

### 3.3

We now turn to the asymptotics on the edge of the limit shape which for the Plancherel measure is described in Theorem 2. Recall that near the edge  $u = 2$  of the limit shape the distance between two consecutive “downs” is of order  $N^{1/6}$  and hence also the probability to observe a “down” in any particular position is of order  $N^{-1/6}$ , where  $N$  is the size of the partition  $\lambda$ . For the Poissonized Plancherel measure  $\mathbb{P}_{\text{PP}}$  the expected size of  $\lambda$  is  $\xi$ . From Theorem 2 we, therefore, expect that

$$\xi^{1/6} \mathbb{K}_{\text{Bessel}} \left( 2\sqrt{\xi} + x\xi^{1/6}, 2\sqrt{\xi} + y\xi^{1/6} \right) \rightarrow \mathbb{K}_{\text{Airy}}(x, y), \quad \xi \rightarrow \infty. \quad (37)$$

We will now use the steepest descent method to deduce (37) from (35). The limit (37) is the key ingredient of the proofs of Theorem 2 given in [5, 17]. The difference between (37) and Theorem 2 is a technical, but rather standard, collection of estimates that allows one to deduce the convergence of joint distributions for Plancherel measures from the convergence of correlation functions of their Poissonizations.

### 3.4

We now start the asymptotic analysis of the integral (35) in the  $u = 2$  regime. What happens in this case is that the two nondegenerate critical points  $e^{\pm i\phi}$  of  $S(z, u)$  coalesce at  $z = 1$  to form one degenerate critical point

$$S(z, 2) = \frac{1}{3}(z - 1)^3 + O((z - 1)^4). \quad (38)$$

Since  $S(z, u)$  appears in the exponent in (35), this starts looking like the Airy integral (10), and, indeed, this is precisely the source of the appearance of the Airy function in (37). In fact, this is the fundamental reason why the Airy function is as ubiquitous in asymptotics as the sine function. The sine function appearance in the steepest descent asymptotics is usually produced by two complex conjugate critical points. When the parameters are tuned so that these two points coalesce, the Airy function enters the scene. This is all very standard and is at length discussed, for example, in [40] and many other places.

The integral (10) converges, but only conditionally, due to ever faster oscillation of the cosine function for large  $s$ . A much better integral can be obtained from (10) by shifting the contour of integration as shown in Figure

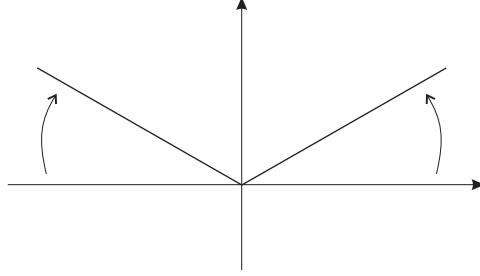


Figure 9: A better contour for the Airy integral (10)

9. The contour in Figure 9 goes along 2 of the 3 downslopes of the graph of  $\Re(is^3)$ , plotted in Figure 10 and commonly known as the monkey saddle. The function  $e^{is^3/3}$  is very rapidly decaying on this contour.

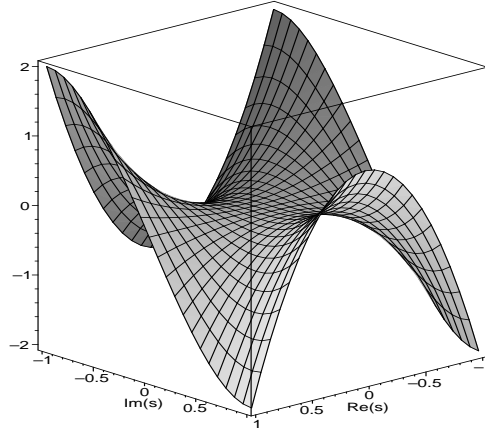


Figure 10: Plot of  $\Re(is^3)$

### 3.5

It is clear that for  $u = 2$  all the action in the integral (35) is happening in the neighborhood of the critical point  $z = w = 1$ , so we want to zoom in on this point. We also want the integration contour in  $z$  to pass through the downslopes of the graph of the real part of (38), while the  $w$ -contour should go along the ridges of the same graph (because of the opposite sign of the  $S\left(w; \frac{y}{\sqrt{\epsilon}}\right)$  term in (35)). We thus deform the integration as shown in the

left half of Figure 11 and introduce the following new integration variables

$$z = 1 + i \frac{z'}{\xi^{1/6}}, \quad w = 1 + i \frac{w'}{\xi^{1/6}}. \quad (39)$$

For large  $\xi$ , the integration contour in  $z'$  and  $w'$  will look like the contour in the right half of Figure 11. The scaling in (39) is chosen so that to make the

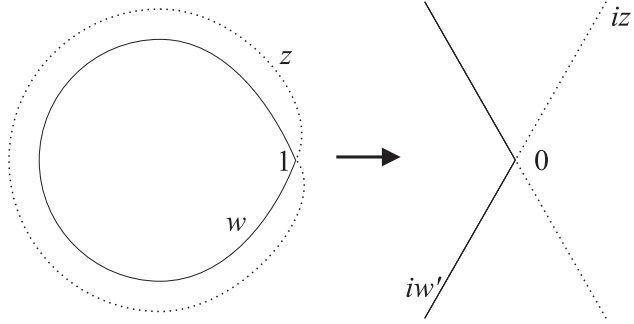


Figure 11: Contours for the edge ( $u = 2$ ) asymptotics

exponent in (35)

$$\sqrt{\xi} S(z, 2) = -\frac{i}{3} (z')^3 + o(1)$$

stay finite as  $\xi \rightarrow \infty$ .

More generally, the leading order expansion of  $S(z, u)$  near  $(z, u) = (1, 2)$  with respect to both arguments is given by

$$\sqrt{\xi} S\left(z, 2 + \frac{x}{\xi^{1/3}}\right) = -ixz' - \frac{i}{3} (z')^3 + o(1), \quad \xi \rightarrow \infty.$$

Plugging this into the integral (35) we obtain the following

$$\xi^{1/6} \mathbb{K}_{\text{Bessel}}\left(2\sqrt{\xi} + x\xi^{1/6}, 2\sqrt{\xi} + y\xi^{1/6}\right) \rightarrow \frac{1}{(2\pi)^2} \iint \frac{e^{-i(z')^3/3 - iz'x + i(w')^3/3 + iw'y}}{i(z' - w')} dz' dw', \quad (40)$$

where the integration contour (rotated by  $90^\circ$ ) is shown in the right half of Figure 11.

### 3.6

It remains to show that

$$\frac{1}{(2\pi)^2} \iint \frac{e^{-i(z')^3/3 - iz'x + i(w')^3/3 + iw'y}}{i(z' - w')} dz' dw' = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}. \quad (41)$$

This is entirely parallel to the argument in Section 2.4 (see the more general discussion in Section 2.2.4 of [28]). We have

$$e^{-iz'x + iw'y} = \frac{i}{x - y} \left( \frac{\partial}{\partial z'} + \frac{\partial}{\partial w'} \right) e^{-iz'x + iw'y}.$$

Inserting this into the LHS of (41) and integrating by parts we obtain

$$\begin{aligned} \frac{1}{(2\pi)^2} \iint \frac{e^{-i(z')^3/3 - iz'x + i(w')^3/3 + iw'y}}{i(z' - w')} dz' dw' = \\ \frac{i}{(2\pi)^2 (x - y)} \iint (z' + w') e^{-i\frac{(z')^3}{3} - iz'x + i\frac{(w')^3}{3} + iw'y} dz' dw'. \end{aligned}$$

The formula (10) translates into

$$\frac{1}{2\pi} \int e^{-i(z')^3/3 - iz'x} dz' = \text{Ai}(x),$$

because our contour of integration in  $z'$  (see Figure 11) is the negative of the contour in Figure 9. It is also clear that

$$\frac{i}{2\pi} \int w' e^{i(w')^3/3 + iw'y} dw' = \text{Ai}'(y).$$

Thus, equation (41) follows.

### 3.7

After finishing these computations, it may be useful to look at them from a more abstract point of view. Above, we tried to be absolutely concrete and explicit, but in fact we used only very general properties of the correlation

kernel (35). Namely, all that was needed was the information about the location of the critical points of the function  $S$  in (35) and the residue of the integrand in (35) at  $z = w$ . The fact that the two critical were complex conjugate, which was important for obtaining the sine function, is an automatic consequence of the fact the integrand is real for real values of the argument. Even less was required for the Airy function asymptotics, the only essential ingredient in which was the coalescence of the two critical points.

This means the same method is applicable to much a wider variety of problems producing the same or equivalent results. This phenomenon, when the final asymptotic result proves to be insensitive to the fine details of the original problem, is known as *universality*. The simplest example of this phenomenon may be the central limit theorem which says that, under very general assumptions, sums of independent, identically distributed, but otherwise arbitrary random variables converge, scaled and centered, to the normal distribution. It is widely believed that the universality principle applies very generally. From a mathematical perspective it is often quite mysterious why this should be the case, but sometimes, like for the central limit theorem, there is a transparent explanation. Similarly, our exact formulas for the correlation functions of the Schur process, combined with the steepest descent argument, offer a simple explanation for the appearance of the sine and Airy kernel.

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